Interval-valued fuzzy graphs

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Abstract

We define the Cartesian product, composition, union and join on interval-valued fuzzy graphs and investigate some of their properties. We also introduce the notion of interval-valued fuzzy complete graphs and present some properties of self complementary and self weak complementary interval-valued fuzzy complete graphs.

Keywords: Interval-valued fuzzy graph, Self complementary, Interval-valued fuzzy complete graph.

Mathematics Subject Classification 2000: 05C99

1 Introduction

In 1975, Zadeh [27] introduced the notion of interval-valued fuzzy sets as an extension of fuzzy sets [26] in which the values of the membership degrees are intervals of numbers in-

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stead of the numbers. Interval-valued fuzzy sets provide a more adequate description of uncertainty than traditional fuzzy sets. It is therefore important to use interval-valued fuzzy sets in applications, such as fuzzy control. One of the computationally most intensive part of fuzzy control is defuzzification [15]. Since interval-valued fuzzy sets are widely studied and used, we describe briefly the work of Gorzalczany on approximate reasoning [10, 11], Roy and Biswas on medical diagnosis [22], Turksen on multivalued logic [25] and Mendel on intelligent control [15].

The fuzzy graph theory as a generalization of Euler's graph theory was first introduced by Rosenfeld [23] in 1975. The fuzzy relations between fuzzy sets were first considered by Rosenfeld and he developed the structure of fuzzy graphs obtaining analogs of several graph theoretical concepts. Later, Bhattacharya [5] gave some remarks on fuzzy graphs, and some operations on fuzzy graphs were introduced by Mordeson and Peng [19]. The complement of a fuzzy graph was defined by Mordeson [18] and further studied by Sunitha and Vijayakumar [24]. Bhutani and Rosenfeld introduced the concept of M-strong fuzzy graphs in [7] and studied some properties. The concept of strong arcs in fuzzy graphs was discussed in [8]. Hongmei and Lianhua gave the definition of interval-valued graph in [12].

In this paper, we define the operations of Cartesian product, composition, union and join on interval-valued fuzzy graphs and investigate some properties. We study isomorphism (resp. weak isomorphism) between interval-valued fuzzy graphs is an equivalence relation (resp. partial order). We introduce the notion of interval-valued fuzzy complete graphs and present some properties of self complementary and self weak complementary interval-valued fuzzy complete graphs.

The definitions and terminologies that we used in this paper are standard. For other notations, terminologies and applications, the readers are referred to [1, 2, 3, 4, 9, 13, 14, 17, 20, 21, 28].

2 Preliminaries

A *graph* is an ordered pair $G^* = (V, E)$, where V is the set of vertices of G^* and E is the set of edges of G^*. Two vertices x and y in a graph G^* are said to be adjacent in G^* if $\{x, y\}$ is in an edge of G^*. (For simplicity an edge $\{x, y\}$ will be denoted by xy.) A *simple graph* is a graph without loops and multiple edges. A *complete graph* is a simple graph in which every pair of distinct vertices is connected by an edge. The complete graph on n vertices has n vertices and $n(n-1)/2$ edges. We will consider only graphs with the finite number of

vertices and edges.

By a *complementary graph* $\overline{G^*}$ of a simple graph G^* we mean a graph having the same vertices as G^* and such that two vertices are adjacent in $\overline{G^*}$ if and only if they are not adjacent in G^*.

An *isomorphism* of graphs G_1^* and G_2^* is a bijection between the vertex sets of G_1^* and G_2^* such that any two vertices v_1 and v_2 of G_1^* are adjacent in G_1^* if and only if $f(v_1)$ and $f(v_2)$ are adjacent in G_2^*. Isomorphic graphs are denoted by $G_1^* \simeq G_2^*$.

Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be two simple graphs, we can construct several new graphs. The first construction called the *Cartesian product* of G_1^* and G_2^* gives a graph $G_1^* \times G_2^* = (V, E)$ with $V = V_1 \times V_2$ and

$$E = \{(x, x_2)(x, y_2)|x \in V_1, x_2y_2 \in E_2\} \cup \{(x_1, z)(y_1, z)|x_1y_1 \in E_1, z \in V_2\}.$$

The *composition* of graphs G_1^* and G_2^* is the graph $G_1^*[G_2^*] = (V_1 \times V_2, E^0)$, where

$$E^0 = E \cup \{(x_1, x_2)(y_1, y_2)|x_1y_1 \in E_1, x_2 \neq y_2\}$$

and E is defined as in $G_1^* \times G_2^*$. Note that $G_1^*[G_2^*] \neq G_2^*[G_1^*]$.

The *union* of graphs G_1^* and G_2^* is defined as $G_1^* \cup G_2^* = (V_1 \cup V_2, E_1 \cup E_2)$.

The *join* of G_1^* and G_2^* is the simple graph $G_1^* + G_2^* = (V_1 \cup V_2, E_1 \cup E_2 \cup E')$, where E' is the set of all edges joining the nodes of V_1 and V_2. In this construction it is assumed that $V_1 \cap V_2 \neq \emptyset$.

By a *fuzzy subset* μ on a set X is mean a map $\mu : X \to [0, 1]$. A map $\nu : X \times X \to [0, 1]$ is called a *fuzzy relation* on X if $\nu(x, y) \leq \min(\mu(x), \mu(y))$ for all $x, y \in X$. A fuzzy relation ν is *symmetric* if $\nu(x, y) = \nu(y, x)$ for all $x, y \in X$.

An *interval number* D is an interval $[a^-, a^+]$ with $0 \leq a^- \leq a^+ \leq 1$. The interval $[a, a]$ is identified with the number $a \in [0, 1]$. $D[0, 1]$ denotes the set of all interval numbers.

For interval numbers $D_1 = [a_1^-, b_1^+]$ and $D_2 = [a_2^-, b_2^+]$, we define

- $\mathrm{rmin}(D_1, D_2) = \mathrm{rmin}([a_1^-, b_1^+], [a_2^-, b_2^+]) = [\min\{a_1^-, a_2^-\}, \min\{b_1^+, b_2^+\}]$,
- $\mathrm{rmax}(D_1, D_2) = \mathrm{rmax}([a_1^-, b_1^+], [a_2^-, b_2^+]) = [\max\{a_1^-, a_2^-\}, \max\{b_1^+, b_2^+\}]$,

- $D_1 + D_2 = [a_1^- + a_2^- - a_1^- \cdot a_2^-, b_1^+ + b_2^+ - b_1^+ \cdot b_2^+]$,
- $D_1 \leq D_2 \Longleftrightarrow a_1^- \leq a_2^-$ and $b_1^+ \leq b_2^+$,
- $D_1 = D_2 \Longleftrightarrow a_1^- = a_2^-$ and $b_1^+ = b_2^+$,
- $D_1 < D_2 \Longleftrightarrow D_1 \leq D_2$ and $D_1 \neq D_2$,
- $kD = k[a_1^-, b_1^+] = [ka_1^-, kb_1^+]$, where $0 \leq k \leq 1$.

Then, $(D[0,1], \leq, \vee, \wedge)$ is a complete lattice with $[0,0]$ as the least element and $[1,1]$ as the greatest.

The *interval-valued fuzzy set* A in V is defined by

$$A = \{(x, [\mu_A^-(x), \mu_A^+(x)]) : x \in V\},$$

where $\mu_A^-(x)$ and $\mu_A^+(x)$ are fuzzy subsets of V such that $\mu_A^-(x) \leq \mu_A^+(x)$ for all $x \in V$. For any two interval-valued sets $A = [\mu_A^-(x), \mu_A^+(x)]$ and $B = [\mu_B^-(x), \mu_B^+(x)])$ in V we define:

- $A \bigcup B = \{(x, \max(\mu_A^-(x), \mu_B^-(x)), \max(\mu_A^+(x), \mu_B^+(x))) : x \in V\}$,
- $A \bigcap B = \{(x, \min(\mu_A^-(x), \mu_B^-(x)), \min(\mu_A^+(x), \mu_B^+(x))) : x \in V\}$.

If $G^* = (V, E)$ is a graph, then by an *interval-valued fuzzy relation* B on a set E we mean an interval-valued fuzzy set such that

$$\mu_B^-(xy) \leq \min(\mu_A^-(x), \mu_A^-(y)),$$

$$\mu_B^+(xy) \leq \min(\mu_A^+(x), \mu_A^+(y))$$

for all $xy \in E$.

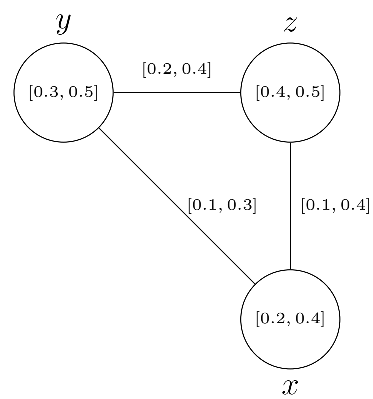
3 Operations on interval-valued fuzzy graphs

Throughout in this paper, G^* is a crisp graph, and G is an interval-valued fuzzy graph.

Definition 3.1. By an *interval-valued fuzzy graph* of a graph $G^* = (V, E)$ we mean a pair $G = (A, B)$, where $A = [\mu_A^-, \mu_A^+]$ is an interval-valued fuzzy set on V and $B = [\mu_B^-, \mu_B^+]$ is an interval-valued fuzzy relation on E.

Example 3.2. Consider a graph $G^* = (V, E)$ such that $V = \{x, y, z\}$, $E = \{xy, yz, zx\}$. Let A be an interval-valued fuzzy set of V and let B be an interval-valued fuzzy set of $E \subseteq V \times V$ defined by

$$A =< (\frac{x}{0.2}, \frac{y}{0.3}, \frac{z}{0.4}), (\frac{x}{0.4}, \frac{y}{0.5}, \frac{z}{0.5}) >,$$
$$B =< (\frac{xy}{0.1}, \frac{yz}{0.2}, \frac{zx}{0.1}), (\frac{xy}{0.3}, \frac{yz}{0.4}, \frac{zx}{0.4}) > .$$



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By routine computations, it is easy to see that $G = (A, B)$ is an interval-valued fuzzy graph of G^*.

Definition 3.3. The *Cartesian product* $G_1 \times G_2$ *of two interval-valued fuzzy graphs* $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ is defined as a pair $(A_1 \times A_1, B_1 \times B_2)$ such that

(i) $$\begin{cases} (\mu_{A_1}^- \times \mu_{A_2}^-)(x_1, x_2) = \min(\mu_{A_1}^-(x_1), \mu_{A_2}^-(x_2)) \\ (\mu_{A_1}^+ \times \mu_{A_2}^+)(x_1, x_2) = \min(\mu_{A_1}^+(x_1), \mu_{A_2}^+(x_2)) \end{cases}$$
for all $(x_1, x_2) \in V$,

(ii) $$\begin{cases} (\mu_{B_1}^- \times \mu_{B_2}^-)((x, x_2)(x, y_2)) = \min(\mu_{A_1}^-(x), \mu_{B_2}^-(x_2 y_2)) \\ (\mu_{B_1}^+ \times \mu_{B_2}^+)((x, x_2)(x, y_2)) = \min(\mu_{A_1}^+(x), \mu_{B_2}^+(x_2 y_2)) \end{cases}$$
for all $x \in V_1$ and $x_2 y_2 \in E_2$,

(iii) $$\begin{cases} (\mu_{B_1}^- \times \mu_{B_2}^-)((x_1, z)(y_1, z)) = \min(\mu_{B_1}^-(x_1 y_1), \mu_{A_2}^-(z)) \\ (\mu_{B_1}^+ \times \mu_{B_2}^+)((x_1, z)(y_1, z)) = \min(\mu_{B_1}^+(x_1 y_1), \mu_{A_2}^+(z)) \end{cases}$$
for all $z \in V_2$ and $x_1 y_1 \in E_1$.

Example 3.4. Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be graphs such that $V_1 = \{a, b\}$, $V_2 = \{c, d\}$, $E_1 = \{ab\}$ and $E_2 = \{cd\}$. Consider two interval-valued fuzzy graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$, where

$$A_1 =< (\frac{a}{0.2}, \frac{b}{0.3}), (\frac{a}{0.4}, \frac{b}{0.5}) >, \qquad B_1 =< \frac{ab}{0.1}, \frac{ab}{0.2} >,$$

$$A_2 =< (\frac{c}{0.1}, \frac{d}{0.2}), (\frac{c}{0.4}, \frac{d}{0.6}) >, \qquad B_2 =< \frac{cd}{0.1}, \frac{cd}{0.3} > .$$

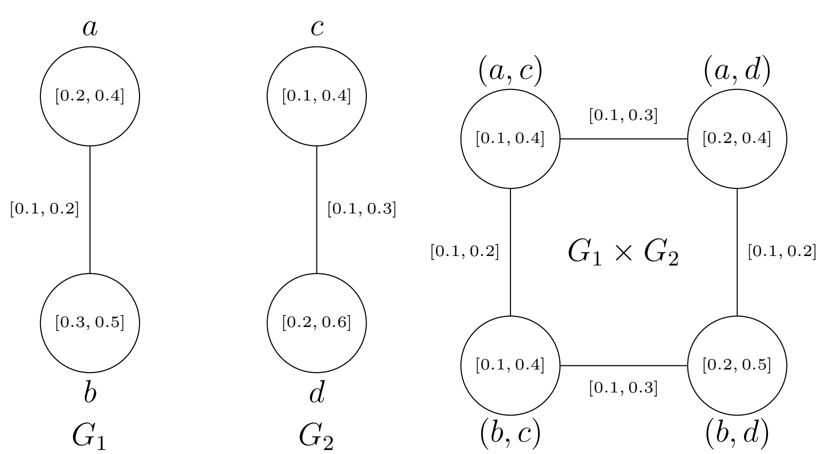
Then, as it is not difficult to verify

$$(\mu_{B_1}^- \times \mu_{B_2}^-)((a, c)(a, d)) = 0.1, \qquad (\mu_{B_1}^+ \times \mu_{B_2}^+)((a, c)(a, d)) = 0.3,$$

$$(\mu_{B_1}^- \times \mu_{B_2}^-)((a, c)(b, c)) = 0.1, \qquad (\mu_{B_1}^+ \times \mu_{B_2}^+)((a, c)(b, c)) = 0.2,$$

$$(\mu_{B_1}^- \times \mu_{B_2}^-)((a, d)(b, d)) = 0.1, \qquad (\mu_{B_1}^+ \times \mu_{B_2}^+)((a, d)(b, d)) = 0.2,$$

$$(\mu_{B_1}^- \times \mu_{B_2}^-)((b, c)(b, d)) = 0.1, \qquad (\mu_{B_1}^+ \times \mu_{B_2}^+)((b, c)(b, d)) = 0.3.$$



By routine computations, it is easy to see that $G_1 \times G_2$ is an interval-valued fuzzy graph of $G_1^* \times G_2^*$.

Proposition 3.5. *The Cartesian product $G_1 \times G_2 = (A_1 \times A_2, B_1 \times B_2)$ of two interval-valued fuzzy graphs of the graphs G_1^* and G_2^* is an interval-valued fuzzy graph of $G_1^* \times G_2^*$.*

Proof. We verify only conditions for $B_1 \times B_2$ because conditions for $A_1 \times A_2$ are obvious.

Let $x \in V_1$, $x_2y_2 \in E_2$. Then

$$\begin{aligned}
(\mu^-_{B_1} \times \mu^-_{B_2})((x,x_2)(x,y_2)) &= \min(\mu^-_{A_1}(x), \mu^-_{B_2}(x_2y_2)) \\
&\leq \min(\mu^-_{A_1}(x), \min(\mu^-_{A_2}(x_2), \mu^-_{A_2}(y_2))) \\
&= \min(\min(\mu^-_{A_1}(x), \mu^-_{A_2}(x_2)), \min(\mu^-_{A_1}(x), \mu^-_{A_2}(y_2))) \\
&= \min((\mu^-_{A_1} \times \mu^-_{A_2})(x,x_2), (\mu^-_{A_1} \times \mu^-_{A_2})(x,y_2)), \\
(\mu^+_{B_1} \times \mu^+_{B_2})((x,x_2)(x,y_2)) &= \min(\mu^+_{A_1}(x), \mu^+_{B_2}(x_2y_2)) \\
&\leq \min(\mu^+_{A_1}(x), \min(\mu^+_{A_2}(x_2), \mu^+_{A_2}(y_2))) \\
&= \min(\min(\mu^+_{A_1}(x), \mu^+_{A_2}(x_2)), \min(\mu^+_{A_1}(x), \mu^+_{A_2}(y_2))) \\
&= \min((\mu^+_{A_1} \times \mu^+_{A_2})(x,x_2), (\mu^+_{A_1} \times \mu^+_{A_2})(x,y_2)).
\end{aligned}$$

Similarly for $z \in V_2$ and $x_1y_1 \in E_1$ we have

$$\begin{aligned}
(\mu^-_{B_1} \times \mu^-_{B_2})((x_1,z)(y_1,z)) &= \min(\mu^-_{B_1}(x_1y_1), \mu^-_{A_2}(z)) \\
&\leq \min(\min(\mu^-_{A_1}(x_1), \mu^-_{A_1}(y_1)), \mu^-_{A_2}(z)) \\
&= \min(\min(\mu^-_{A_1}(x), \mu^-_{A_2}(z)), \min(\mu^-_{A_1}(y_1), \mu^-_{A_2}(z))) \\
&= \min((\mu^-_{A_1} \times \mu^-_{A_2})(x_1,z), (\mu^-_{A_1} \times \mu^-_{A_2})(y_1,z)), \\
(\mu^+_{B_1} \times \mu^+_{B_2})((x_1,z)(y_1,z)) &= \min(\mu^+_{B_1}(x_1y_1), \mu^+_{A_2}(z)) \\
&\leq \min(\min(\mu^+_{A_1}(x_1), \mu^+_{A_1}(y_1)), \mu^+_{A_2}(z)) \\
&= \min(\min(\mu^+_{A_1}(x), \mu^+_{A_2}(z)), \min(\mu^+_{A_1}(y_1), \mu^+_{A_2}(z))) \\
&= \min((\mu^+_{A_1} \times \mu^+_{A_2})(x_1,z), (\mu^+_{A_1} \times \mu^+_{A_2})(y_1,z)).
\end{aligned}$$

This completes the proof. $\square$

Definition 3.6. The *composition* $G_1[G_2] = (A_1 \circ A_2, B_1 \circ B_2)$ *of two interval-valued fuzzy graphs* G_1 and G_2 of the graphs G_1^* and G_2^* is defined as follows:

(i) $\begin{cases} (\mu^-_{A_1} \circ \mu^-_{A_2})(x_1,x_2) = \min(\mu^-_{A_1}(x_1), \mu^-_{A_2}(x_2)) \\ (\mu^+_{A_1} \circ \mu^+_{A_2})(x_1,x_2) = \min(\mu^+_{A_1}(x_1), \mu^+_{A_2}(x_2)) \end{cases}$

for all $(x_1,x_2) \in V$,

(ii) $\begin{cases} (\mu^-_{B_1} \circ \mu^-_{B_2})((x,x_2)(x,y_2)) = \min(\mu^-_{A_1}(x), \mu^-_{B_2}(x_2y_2)) \\ (\mu^+_{B_1} \circ \mu^+_{B_2})((x,x_2)(x,y_2)) = \min(\mu^+_{A_1}(x), \mu^+_{B_2}(x_2y_2)) \end{cases}$

for all $x \in V_1$ and $x_2y_2 \in E_2$,

(iii) $\begin{cases} (\mu^-_{B_1} \circ \mu^-_{B_2})((x_1, z)(y_1, z)) = \min(\mu^-_{B_1}(x_1y_1), \mu^-_{A_2}(z)) \\ (\mu^+_{B_1} \circ \mu^+_{B_2})((x_1, z)(y_1, z)) = \min(\mu^+_{B_1}(x_1y_1), \mu^+_{A_2}(z)) \end{cases}$

for all $z \in V_2$ and $x_1y_1 \in E_1$,

(iv) $\begin{cases} (\mu^-_{B_1} \circ \mu^-_{B_2})((x_1, x_2)(y_1, y_2)) = \min(\mu^-_{A_2}(x_2), \mu^-_{A_2}(y_2), \mu^-_{B_1}(x_1y_1)) \\ (\mu^+_{B_1} \circ \mu^+_{B_2})((x_1, x_2)(y_1, y_2)) = \min(\mu^+_{A_2}(x_2), \mu^+_{A_2}(y_2), \mu^+_{B_1}(x_1y_1)) \end{cases}$

for all $(x_1, x_2)(y_1, y_2) \in E^0 - E$.

Example 3.7. Let G_1^* and G_2^* be as in the previous example. Consider two interval-valued fuzzy graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ defined by

$$A_1 = < (\frac{a}{0.2}, \frac{b}{0.3}), (\frac{a}{0.5}, \frac{b}{0.5}) >, \qquad B_1 = < \frac{ab}{0.2}, \frac{ab}{0.4} >,$$

$$A_2 = < (\frac{c}{0.1}, \frac{d}{0.3}), (\frac{c}{0.4}, \frac{d}{0.6}) >, \qquad B_2 = < \frac{cd}{0.1}, \frac{cd}{0.3} > .$$

Then we have

$$(\mu^-_{B_1} \circ \mu^-_{B_2})((a, c)(a, d)) = 0.2, \qquad (\mu^+_{B_1} \circ \mu^+_{B_2})((a, c)(a, d)) = 0.3,$$

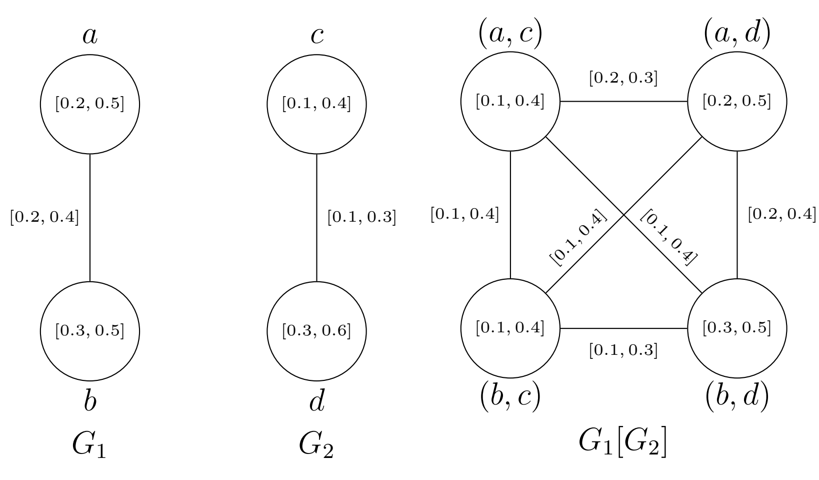
$$(\mu^-_{B_1} \circ \mu^-_{B_2})((b, c)(b, d)) = 0.1, \qquad (\mu^+_{B_1} \circ \mu^+_{B_2})((b, c)(b, d)) = 0.3,$$

$$(\mu^-_{B_1} \circ \mu^-_{B_2})((a, c)(b, c)) = 0.1, \qquad (\mu^+_{B_1} \circ \mu^+_{B_2})((a, c)(b, c)) = 0.4,$$

$$(\mu^-_{B_1} \circ \mu^-_{B_2})((a, d)(b, d)) = 0.2, \qquad (\mu^+_{B_1} \circ \mu^+_{B_2})((a, d)(b, d)) = 0.4,$$

$$(\mu^-_{B_1} \circ \mu^-_{B_2})((a, c)(b, d)) = 0.1, \qquad (\mu^+_{B_1} \circ \mu^+_{B_2})((a, c)(b, d)) = 0.4,$$

$$(\mu^-_{B_1} \circ \mu^-_{B_2})((b, c)(a, d)) = 0.1, \qquad (\mu^+_{B_1} \circ \mu^+_{B_2})((b, c)(a, d)) = 0.4.$$



By routine computations, it is easy to see that $G_1[G_2] = (A_1 \circ A_2, B_1 \circ B_2)$ is an interval-valued fuzzy graph of $G_1^*[G_2^*]$.

Proposition 3.8. *The composition $G_1[G_2]$ of interval-valued fuzzy graphs G_1 and G_2 of G_1^* and G_2^* is an interval-valued fuzzy graph of $G_1^*[G_2^*]$.*

Proof. Similarly as in the previous proof we verify the conditions for $B_1 \circ B_2$ only.

In the case $x \in V_1$, $x_2y_2 \in E_2$, according to (ii) we obtain

$$\begin{aligned}
(\mu^-_{B_1} \circ \mu^-_{B_2})((x, x_2)(x, y_2)) &= \min(\mu^-_{A_1}(x), \mu^-_{B_2}(x_2y_2)) \\
&\leq \min(\mu^-_{A_1}(x), \min(\mu^-_{A_2}(x_2), \mu^-_{A_2}(y_2))) \\
&= \min(\min(\mu^-_{A_1}(x), \mu^-_{A_2}(x_2)), \min(\mu^-_{A_1}(x), \mu^-_{A_2}(y_2))) \\
&= \min((\mu^-_{A_1} \circ \mu^-_{A_2})(x, x_2), (\mu^-_{A_1} \circ \mu^-_{A_2})(x, y_2)), \\
(\mu^+_{B_1} \circ \mu^+_{B_2})((x, x_2)(x, y_2)) &= \min(\mu^+_{A_1}(x), \mu^+_{B_2}(x_2y_2)) \\
&\leq \min(\mu^+_{A_1}(x), \min(\mu^+_{A_2}(x_2), \mu^+_{A_2}(y_2))) \\
&= \min(\min(\mu^+_{A_1}(x), \mu^+_{A_2}(x_2)), \min(\mu^+_{A_1}(x), \mu^+_{A_2}(y_2))) \\
&= \min((\mu^+_{A_1} \circ \mu^+_{A_2})(x, x_2), (\mu^+_{A_1} \circ \mu^+_{A_2})(x, y_2)).
\end{aligned}$$

In the case $z \in V_2$, $x_1y_1 \in E_1$ the proof is similar.

In the case $(x_1, x_2)(y_1, y_2) \in E^0 - E$ we have $x_1y_1 \in E_1$ and $x_2 \neq y_2$, which according to (iv) implies

$$\begin{aligned}
(\mu^-_{B_1} \circ \mu^-_{B_2})((x_1, x_2)(y_1, y_2)) &= \min(\mu^-_{A_2}(x_2), \mu^-_{A_2}(y_2), \mu^-_{B_1}(x_1y_1)) \\
&\leq \min(\mu^-_{A_2}(x_2), \mu^-_{A_2}(y_2), \min(\mu^-_{A_1}(x_1), \mu^-_{A_1}(y_1))) \\
&= \min(\min(\mu^-_{A_1}(x_1), \mu^-_{A_2}(x_2)), \min(\mu^-_{A_1}(y_1), \mu^-_{A_2}(y_2))) \\
&= \min((\mu^-_{A_1} \circ \mu^-_{A_2})(x_1, x_2), (\mu^-_{A_1} \circ \mu^-_{A_2})(y_1, y_2)), \\
(\mu^+_{B_1} \circ \mu^+_{B_2})((x_1, x_2)(y_1, y_2)) &= \min(\mu^+_{A_2}(x_2), \mu^+_{A_2}(y_2), \mu^+_{B_1}(x_1y_1)) \\
&\leq \min(\mu^+_{A_2}(x_2), \mu^+_{A_2}(y_2), \min(\mu^+_{A_1}(x_1), \mu^+_{A_1}(y_1))) \\
&= \min(\min(\mu^+_{A_1}(x_1), \mu^+_{A_2}(x_2)), \min(\mu^+_{A_1}(y_1), \mu^+_{A_2}(y_2))) \\
&= \min((\mu^+_{A_1} \circ \mu^+_{A_2})(x_1, x_2), (\mu^+_{A_1} \circ \mu^+_{A_2})(y_1, y_2)).
\end{aligned}$$

This completes the proof. $\square$

Definition 3.9. The *union* $G_1 \cup G_2 = (A_1 \cup A_2, B_1 \cup B_2)$ *of two interval-valued fuzzy graphs* G_1 and G_2 of the graphs G_1^* and G_2^* is defined as follows:

$$\text{(A)} \quad \begin{cases} (\mu^-_{A_1} \cup \mu^-_{A_2})(x) = \mu^-_{A_1}(x) \quad \text{if } x \in V_1 \text{ and } x \notin V_2, \\ (\mu^-_{A_1} \cup \mu^-_{A_2})(x) = \mu^-_{A_2}(x) \quad \text{if } x \in V_2 \text{ and } x \notin V_1, \\ (\mu^-_{A_1} \cup \mu^-_{A_2})(x) = \max(\mu^-_{A_1}(x), \mu^-_{A_2}(x)) \ \text{ if } x \in V_1 \cap V_2, \end{cases}$$

$$\text{(B)} \quad \begin{cases} (\mu^+_{A_1} \cup \mu^+_{A_2})(x) = \mu^+_{A_1}(x) \quad \text{if } x \in V_1 \text{ and } x \notin V_2, \\ (\mu^+_{A_1} \cup \mu^+_{A_2})(x) = \mu^+_{A_2}(x) \quad \text{if } x \in V_2 \text{ and } x \notin V_1, \\ (\mu^+_{A_1} \cup \mu^+_{A_2})(x) = \max(\mu^+_{A_1}(x), \mu^+_{A_2}(x)) \ \text{ if } x \in V_1 \cap V_2, \end{cases}$$

$$\text{(C)} \quad \begin{cases} (\mu^-_{B_1} \cup \mu^-_{B_2})(xy) = \mu^-_{B_1}(xy) \quad \text{if } xy \in E_1 \text{ and } xy \notin E_2, \\ (\mu^-_{B_1} \cup \mu^-_{B_2})(xy) = \mu^-_{B_2}(xy) \quad \text{if } xy \in E_2 \text{ and } xy \notin E_1, \\ (\mu^-_{B_1} \cup \mu^-_{B_2})(xy) = \max(\mu^-_{B_1}(xy), \mu^-_{B_2}(xy)) \ \text{ if } xy \in E_1 \cap E_2, \end{cases}$$

$$\text{(D)} \quad \begin{cases} (\mu^+_{B_1} \cup \mu^+_{B_2})(xy) = \mu^+_{B_1}(xy) \quad \text{if } xy \in E_1 \text{ and } xy \notin E_2, \\ (\mu^+_{B_1} \cup \mu^+_{B_2})(xy) = \mu^+_{B_2}(xy) \quad \text{if } xy \in E_2 \text{ and } xy \notin E_1, \\ (\mu^+_{B_1} \cup \mu^+_{B_2})(xy) = \max(\mu^+_{B_1}(xy), \mu^+_{B_2}(xy)) \ \text{ if } xy \in E_1 \cap E_2. \end{cases}$$

Example 3.10. Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be graphs such that $V_1 = \{a, b, c, d, e\}$, $E_1 = \{ab, bc, be, ce, ad, ed\}$, $V_2 = \{a, b, c, d, f\}$ and $E_2 = \{ab, bc, cf, bf, bd\}$. Consider two interval-valued fuzzy graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ defined by

$$A_1 =< (\frac{a}{0.2}, \frac{b}{0.4}, \frac{c}{0.3}, \frac{d}{0.3}, \frac{e}{0.2}), (\frac{a}{0.4}, \frac{b}{0.5}, \frac{c}{0.6}, \frac{d}{0.7}, \frac{e}{0.6}) >,$$

$$B_1 =< (\frac{ab}{0.1}, \frac{bc}{0.2}, \frac{ce}{0.1}, \frac{be}{0.2}, \frac{ad}{0.1}, \frac{de}{0.1}), (\frac{ab}{0.3}, \frac{bc}{0.4}, \frac{ce}{0.5}, \frac{be}{0.5}, \frac{ad}{0.3}, \frac{de}{0.6}) >,$$

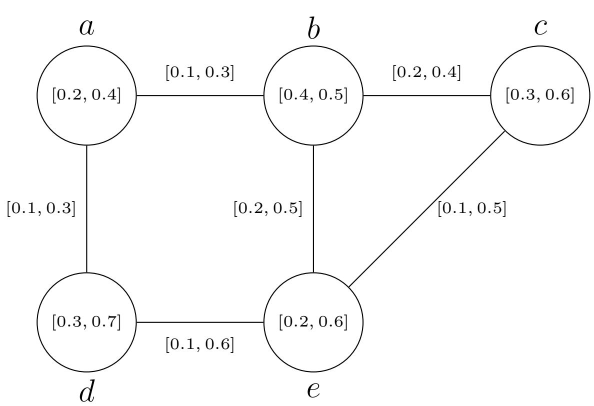
$$A_2 =< (\frac{a}{0.2}, \frac{b}{0.2}, \frac{c}{0.3}, \frac{d}{0.2}, \frac{f}{0.4}), (\frac{a}{0.4}, \frac{b}{0.5}, \frac{c}{0.6}, \frac{d}{0.6}, \frac{f}{0.6}) >,$$

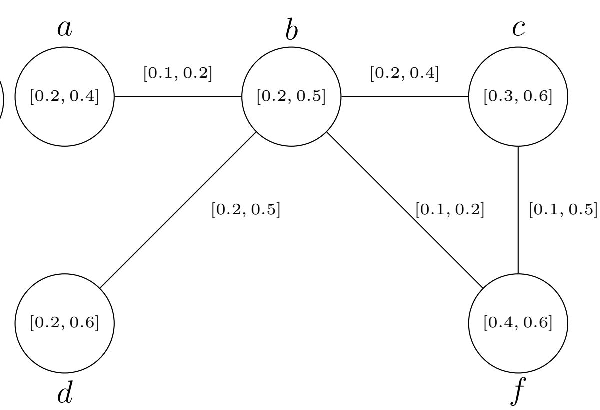
$$B_2 =< (\frac{ab}{0.1}, \frac{bc}{0.2}, \frac{cf}{0.1}, \frac{bf}{0.1}, \frac{bd}{0.2}), (\frac{ab}{0.2}, \frac{bc}{0.4}, \frac{cf}{0.5}, \frac{bf}{0.2}, \frac{bd}{0.5}) > .$$

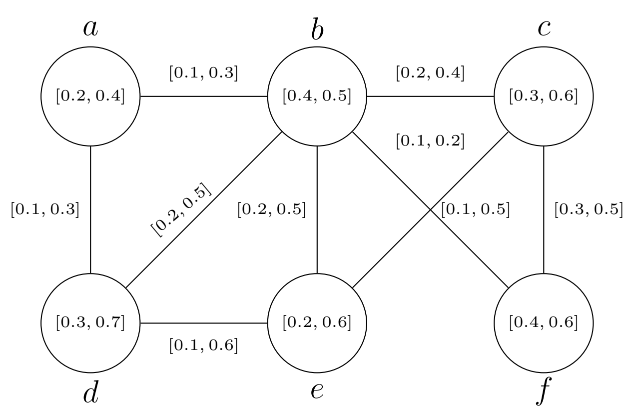
Then, according to the above definition:

$$
\begin{array}{llll}
 & (\mu^-_{A_1} \cup \mu^-_{A_2})(a) = 0.2, & (\mu^-_{A_1} \cup \mu^-_{A_2})(b) = 0.4, & \\
 & (\mu^-_{A_1} \cup \mu^-_{A_2})(c) = 0.3, & (\mu^-_{A_1} \cup \mu^-_{A_2})(d) = 0.3, & \\
 & (\mu^-_{A_1} \cup \mu^-_{A_2})(e) = 0.2, & (\mu^-_{A_1} \cup \mu^-_{A_2})(f) = 0.4, & \\
 & (\mu^+_{A_1} \cup \mu^+_{A_2})(a) = 0.4, & (\mu^+_{A_1} \cup \mu^+_{A_2})(b) = 0.5, & \\
(\mu^+_{A_1} \cup \mu^+_{A_2})(c) = 0.6, & (\mu^+_{A_1} \cup \mu^+_{A_2})(d) = 0.7, & (\mu^+_{A_1} \cup \mu^+_{A_2})(e) = 0.1, & (\mu^+_{A_1} \cup \mu^+_{A_2})(f) = 0.6, \\
(\mu^-_{B_1} \cup \mu^-_{B_2})(ab) = 0.1, & (\mu^-_{B_1} \cup \mu^-_{B_2})(bc) = 0.2, & (\mu^-_{B_1} \cup \mu^-_{B_2})(ce) = 0.1, & (\mu^-_{B_1} \cup \mu^-_{B_2})(be) = 0.2, \\
(\mu^-_{B_1} \cup \mu^-_{B_2})(ad) = 0.1, & (\mu^-_{B_1} \cup \mu^-_{B_2})(de) = 0.1, & (\mu^-_{B_1} \cup \mu^-_{B_2})(bd) = 0.2, & (\mu^-_{B_1} \cup \mu^-_{B_2})(bf) = 0.1, \\
(\mu^+_{B_1} \cup \mu^+_{B_3})(ab) = 0.3, & (\mu^+_{B_1} \cup \mu^+_{B_2})(bc) = 0.4, & (\mu^+_{B_1} \cup \mu^+_{B_2})(ce) = 0.5, & (\mu^+_{B_1} \cup \mu^+_{B_2})(be) = 0.5, \\
(\mu^+_{B_1} \cup \mu^+_{B_2})(ad) = 0.3, & (\mu^+_{B_1} \cup \mu^+_{B_2})(de) = 0.6, & (\mu^+_{B_1} \cup \mu^+_{B_2})(bd) = 0.5, & (\mu^+_{B_1} \cup \mu^+_{B_2})(bf) = 0.2.
\end{array}
$$

G_1 G_2







$G_1 \cup G_2$

Clearly, $G_1 \cup G_2 = (A_1 \cup A_2, B_1 \cup B_2)$ is an interval-valued fuzzy graph of the graph $G_1^* \cup G_2^*$.

Proposition 3.11. *The union of two interval-valued fuzzy graphs is an interval-valued fuzzy graph.*

Proof. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be interval-valued fuzzy graphs of G_1^* and G_2^*, respectively. We prove that $G_1 \cup G_2 = (A_1 \cup A_2, B_1 \cup B_2)$ is an interval-valued fuzzy graph of the graph $G_1^* \cup G_2^*$. Since all conditions for $A_1 \cup A_2$ are automatically satisfied we verify only conditions for $B_1 \cup B_2$.

At first we consider the case when $xy \in E_1 \cap E_2$. Then

$$\begin{array}{rcl}
(\mu^-_{B_1} \cup \mu^-_{B_2})(xy) & = & \max(\mu^-_{B_1}(xy), \mu^-_{B_2}(xy)) \\
& \leq & \max(\min(\mu^-_{A_1}(x), \mu^-_{A_1}(y)), \min(\mu^-_{A_2}(x), \mu^-_{A_2}(y))) \\
& = & \min(\max(\mu^-_{A_1}(x), \mu^-_{A_2}(x)), \max(\mu^-_{A_1}(y), \mu^-_{A_2}(y))) \\
& = & \min((\mu^-_{A_1} \cup \mu^-_{A_2})(x), (\mu^-_{A-1} \cup \mu^-_{A_2})(y)), \\
(\mu^+_{B_1} \cup \mu^+_{B_2})(xy) & = & \max(\mu^+_{B_1}(xy), \mu^+_{B_2}(xy)) \\
& \leq & \max(\min(\mu^+_{A_1}(x), \mu^+_{A_1}(y)), \min(\mu^+_{A_2}(x), \mu^+_{A_2}(y))) \\
& = & \min(\max(\mu^+_{A_1}(x), \mu^+_{A_2}(x)), \max(\mu^+_{A_1}(y), \mu^+_{A_2}(y))) \\
& = & \min((\mu^+_{A_1} \cup \mu^+_{A_2})(x), (\mu^+_{A-1} \cup \mu^+_{A_2})(y)).
\end{array}$$

If $xy \in E_1$ and $xy \notin E_2$, then

$$(\mu^-_{B_1} \cup \mu^-_{B_2})(xy) \leq \min((\mu^-_{A_1} \cup \mu^-_{A_2})(x), (\mu^-_{A_1} \cup \mu^-_{A_2})(y)),$$

$$(\mu^+_{B_1} \cup \mu^+_{B_2})(xy) \leq \min((\mu^+_{A_1} \cup \mu^+_{A_2})(x), (\mu^+_{A_1} \cup \mu^+_{A_2})(y)).$$

If $xy \in E_2$ and $xy \notin E_1$, then

$$(\mu^-_{B_1} \cup \mu^-_{B_2})(xy) \leq \min((\mu^-_{A_1} \cup \mu^-_{A_2})(x), (\mu^-_{A_1} \cup \mu^-_{A_2})(y)),$$

$$(\mu^+_{B_1} \cup \mu^+_{B_2})(xy) \leq \min((\mu^+_{A_1} \cup \mu^+_{A_2})(x), (\mu^+_{A_1} \cup \mu^+_{A_2})(y)).$$

This completes the proof. □

Definition 3.12. The *join* $G_1 + G_2 = (A_1 + A_2, B_1 + B_2)$ *of two interval-valued fuzzy graphs* G_1 and G_2 of the graphs G_1^* and G_2^* is defined as follows:

(A) $\begin{cases} (\mu^-_{A_1} + \mu^-_{A_2})(x) = (\mu^-_{A_1} \cup \mu^-_{A_2})(x) \\ (\mu^+_{A_1} + \mu^+_{A_2})(x) = (\mu^+_{A_1} \cup \mu^+_{A_2})(x) \end{cases}$

if $x \in V_1 \cup V_2$,

(B) $\begin{cases} (\mu^-_{B_1} + \mu^-_{B_2})(xy) = (\mu^-_{B_1} \cup \mu^-_{B_2})(xy) \\ (\mu^+_{B_1} + \mu^+_{B_2})(xy) = (\mu^+_{B_1} \cup \mu^+_{B_2})(xy) \end{cases}$

if $xy \in E_1 \cap E_2$,

(C) $\begin{cases} (\mu^-_{B_1} + \mu^-_{B_2})(xy) = \min(\mu^-_{A_1}(x), \mu^-_{A_2}(y)) \\ (\mu^+_{B_1} + \mu^+_{B_2})(xy) = \min(\mu^+_{A_1}(x), \mu^+_{A_2}(y)) \end{cases}$

if $xy \in E'$, where E' is the set of all edges joining the nodes of V_1 and V_2.

Proposition 3.13. *The join of interval-valued fuzzy graphs is an interval-valued fuzzy graph.*

Proof. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be interval-valued fuzzy graphs of G_1^* and G_2^*, respectively. We prove that $G_1+G_2 = (A_1+A_2, B_1+B_2)$ is an interval-valued fuzzy graph of the graph $G_1^* + G_2^*$. In view of Proposition 3.11 is sufficient to verify the case when $xy \in E'$. In this case we have

$$\begin{aligned}
(\mu^-_{B_1} + \mu^-_{B_2})(xy) &= \min(\mu^-_{A_1}(x), \mu^-_{A_2}(y)) \\
&\leq \min((\mu^-_{A_1} \cup \mu^-_{A_2})(x), (\mu^-_{A_1} \cup \mu^-_{A_2})(y)) \\
&= \min((\mu^-_{A_1} + \mu^-_{A_2})(x), (\mu^-_{A_1} + \mu^-_{A_2})(y)), \\
(\mu^+_{B_1} + \mu^+_{B_2})(xy) &= \min(\mu^+_{A_1}(x), \mu^+_{A_2}(y)) \\
&\leq \min((\mu^+_{A_1} \cup \mu^+_{A_2})(x), (\mu^+_{A_1} \cup \mu^+_{A_2})(y)) \\
&= \min((\mu^+_{A_1} + \mu^+_{A_2})(x), (\mu^+_{A_1} + \mu^+_{A_2})(y)).
\end{aligned}$$

This completes the proof. □

Proposition 3.14. *Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be crisp graphs with $V_1 \cap V_2 = \emptyset$. Let A_1, A_2, B_1 and B_2 be interval-valued fuzzy subsets of V_1, V_2, E_1 and E_2, respectively. Then $G_1 \cup G_2 = (A_1 \cup A_2, B_1 \cup B_2)$ is an interval-valued fuzzy graph of $G_1^* \cup G_2^*$ if and only if $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ are interval-valued fuzzy graphs of G_1^* and G_2^*, respectively.*

Proof. Suppose that $G_1 \cup G_2 = (A_1 \cup A_2, B_1 \cup B_2)$ is an interval-valued fuzzy graph of $G_1^* \cup G_2^*$.

Let $xy \in E_1$. Then $xy \notin E_2$ and $x, y \in V_1 - V_2$. Thus

$$\begin{aligned}
\mu^-_{B_1}(xy) &= (\mu^-_{B_1} \cup \mu^-_{B_2})(xy) \\
&\leq \min((\mu^-_{A_1} \cup \mu^-_{A_2})(x), (\mu^-_{A_1} \cup \mu^-_{A_2})(y)) \\
&= \min(\mu^-_{A_1}(x), \mu^-_{A_1}(y)), \\
\mu^+_{B_1}(xy) &= (\mu^+_{B_1} \cup \mu^+_{B_2})(xy) \\
&\leq \min((\mu^+_{A_1} \cup \mu^+_{A_2})(x), (\mu^+_{A_1} \cup \mu^+_{A_2})(y)) \\
&= \min(\mu^+_{A_1}(x), \mu^+_{A_1}(y)).
\end{aligned}$$

This shows that $G_1 = (A_1, B_1)$ is an interval-valued fuzzy graph. Similarly, we can show that $G_2 = (A_2, B_2)$ is an interval-valued fuzzy graph.

The converse statement is given by Proposition 3.11. □

As a consequence of Propositions 3.13 and 3.14 we obtain

Proposition 3.15. *Let $G^*_1 = (V_1, E_1)$ and $G^*_2 = (V_2, E_2)$ be crisp graphs and let $V_1 \cap V_2 = \emptyset$. Let A_1, A_2, B_1 and B_2 be interval-valued fuzzy subsets of V_1, V_2, E_1 and E_2, respectively. Then $G_1 + G_2 = (A_1 + A_2, B_1 + B_2)$ is an interval-valued fuzzy graph of G^* if and only if $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ are interval-valued fuzzy graphs of G^*_1 and G^*_2, respectively.*

4 Isomorphisms of interval-valued fuzzy graphs

In this section we characterize various types of (weak) isomorphisms of interval valued graphs.

Definition 4.1. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two interval-valued fuzzy graphs. A *homomorphism* $f : G_1 \to G_2$ is a mapping $f : V_1 \to V_2$ such that

(a) $\mu^-_{A_1}(x_1) \leq \mu^-_{A_2}(f(x_1))$, $\quad \mu^+_{A_1}(x_1) \leq \mu^+_{A_2}(f(x_1))$,

(b) $\mu^-_{B_1}(x_1y_1) \leq \mu^-_{B_2}(f(x_1)f(y_1))$, $\quad \mu^+_{B_1}(x_1y_1) \leq \mu^+_{B_2}(f(x_1)f(y_1))$

for all $x_1 \in V_1$, $x_1y_1 \in E_1$.

A bijective homomorphism with the property

(c) $\mu^-_{A_1}(x_1) = \mu^-_{A_2}(f(x_1)), \quad \mu^+_{A_1}(x_1) = \mu^+_{A_2}(f(x_1)),$

is called a *weak isomorphism*. A weak isomorphism preserves the weights of the nodes but not necessarily the weights of the arcs.

A bijective homomorphism preserving the weights of the arcs but not necessarily the weights of nodes, i.e., a bijective homomorphism $f : G_1 \to G_2$ such that

(d) $\mu^-_{B_1}(x_1y_1) = \mu^-_{B_2}(f(x_1)f(y_1)),\ \mu^+_{B_1}(x_1y_1) = \mu^+_{B_2}(f(x_1)f(y_1))$

for all $x_1y_1 \in V_1$ is called a *weak co-isomorphism*.

A bijective mapping $f : G_1 \to G_2$ satisfying (c) and (d) is called an *isomorphism*.

Example 4.2. Consider graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ such that $V_1 = \{a_1, b_1\}$, $V_2 = \{a_2, b_2\}$, $E_1 = \{a_1b_1\}$ and $E_2 = \{a_2b_2\}$. Let A_1, A_2, B_1 and B_2 be interval-valued fuzzy subsets defined by

$$A_1 =< (\frac{a_1}{0.2}, \frac{b_1}{0.3}), (\frac{a_1}{0.5}, \frac{b_1}{0.6}) >, \quad B_1 =< \frac{a_1b_1}{0.1}, \frac{a_1b_1}{0.3} >,$$
$$A_2 =< (\frac{a_2}{0.3}, \frac{b_2}{0.2}), (\frac{a_2}{0.6}, \frac{b_2}{0.5}) >, \quad B_2 =< \frac{a_2b_2}{0.1}, \frac{a_2b_2}{0.4} > .$$

Then, as it is easy to see, $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ are interval-valued fuzzy graphs of G_1^* and G_2^*, respectively. The map $f : V_1 \to V_2$ defined by $f(a_1) = b_2$ and $f(b_1) = a_2$ is a weak isomorphism but it is not an isomorphism.

Example 4.3. Let G_1^* and G_2^* be as in the previous example and let A_1, A_2, B_1 and B_2 be interval-valued fuzzy subsets defined by

$$A_1 =< (\frac{a_1}{0.2}, \frac{b_1}{0.3}), (\frac{a_1}{0.4}, \frac{b_1}{0.5}) >, \quad B_1 =< \frac{a_1b_1}{0.1}, \frac{a_1b_1}{0.3} >,$$
$$A_2 =< (\frac{a_2}{0.4}, \frac{b_2}{0.3}), (\frac{a_2}{0.5}, \frac{b_2}{0.6}) >, \quad B_2 =< \frac{a_2b_2}{0.1}, \frac{a_2b_2}{0.3} > .$$

Then $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ are interval-valued fuzzy graphs of G_1^* and G_2^*, respectively. The map $f : V_1 \to V_2$ defined by $f(a_1) = b_2$ and $f(b_1) = a_2$ is a weak co-isomorphism but it is not an isomorphism.

Proposition 4.4. *An isomorphism between interval-valued fuzzy graphs is an equivalence relation.*

Problem. Prove or disprove that weak isomorphism (co-isomorphism) between interval-valued fuzzy graphs is a partial ordering relation.

5 Interval-valued fuzzy complete graphs

Definition 5.1. An interval-valued fuzzy graph $G = (A, B)$ is called *complete* if

$$\mu_B^-(xy) = \min(\mu_A^-(x), \mu_A^-(y)) \quad \text{and} \quad \mu_B^+(xy) = \min(\mu_A^+(x), \mu_A^+(y)) \quad \text{for all} \ \ xy \in E.$$

Example 5.2. Consider a graph $G^* = (V, E)$ such that $V = \{x, y, z\}$, $E = \{xy, yz, zx\}$. If A and B are interval-valued fuzzy subset defined by

$$A =< (\frac{x}{0.2}, \frac{y}{0.3}, \frac{z}{0.4}), (\frac{x}{0.4}, \frac{y}{0.5}, \frac{z}{0.5}) >,$$
$$B =< (\frac{xy}{0.2}, \frac{yz}{0.3}, \frac{zx}{0.2}), (\frac{xy}{0.4}, \frac{yz}{0.5}, \frac{zx}{0.4}) >,$$

then $G = (A, B)$ is an interval-valued fuzzy complete graph of G^*.

As a consequence of Proposition 3.8 we obtain

Proposition 5.3. *If $G = (A, B)$ be an interval-valued fuzzy complete graph, then also $G[G]$ is an interval-valued fuzzy complete graph.*

Definition 5.4. The *complement* of an interval-valued fuzzy complete graph $G = (A, B)$ of $G^* = (V, E)$ is an interval-valued fuzzy complete graph $\overline{G} = (\overline{A}, \overline{B})$ on $\overline{G^*} = (V, \overline{E})$, where $\overline{A} = A = [\mu_A^-, \mu_A^+]$ and $\overline{B} = [\overline{\mu^-}_B, \overline{\mu^+}_B]$ is defined by

$$\overline{\mu_B^-}(xy) = \begin{cases} 0 & \text{if} \ \ \mu_B^-(xy) > 0, \\ \min(\mu_A^-(x), \mu_A^-(y)) & \text{if} \ \ \text{if} \ \mu_B^-(xy) = 0, \end{cases}$$

$$\overline{\mu_B^+}(xy) = \begin{cases} 0 & \text{if} \ \ \mu_B^+(xy) > 0, \\ \min(\mu_A^+(x), \mu_A^+(y)) & \text{if} \ \ \text{if} \ \mu_B^+(xy) = 0. \end{cases}$$

Definition 5.5. An interval-valued fuzzy complete graph $G = (A, B)$ is called *self complementary* if $\overline{\overline{G}} = G$.

Example 5.6. Consider a graph $G^* = (V, E)$ such that $V = \{a, b, c\}$, $E = \{ab, bc\}$. Then an interval-valued fuzzy graph $G = (A, B)$, where

$$A =< (\frac{a}{0.1}, \frac{b}{0.2}, \frac{c}{0.3}), (\frac{a}{0.3}, \frac{b}{0.4}, \frac{c}{0.5}) >,$$
$$B =< (\frac{ab}{0.1}, \frac{bc}{0.2}), (\frac{ab}{0.3}, \frac{bc}{0.4}) >,$$

is self complementary.

Proposition 5.7. *In a self complementary interval-valued fuzzy complete graph $G = (A, B)$ we have*

a) $\sum_{x \neq y} \mu_B^-(xy) = \sum_{x \neq y} \min(\mu_A^-(x), \mu_A^-(y))$,

b) $\sum_{x \neq y} \mu_B^+(xy) = \sum_{x \neq y} \min(\mu_A^+(x), \mu_A^+(y))$.

Proof. Let $G = (A, B)$ be a self complementary interval-valued fuzzy complete graph. Then there exists an automorphism $f : V \to V$ such that $\mu_A^-(f(x)) = \mu_A^-(x)$, $\mu_A^+(f(x)) = \mu_A^+(x)$, $\overline{\mu_B^-}(f(x)f(y)) = \mu_B^-(xy)$ and $\overline{\mu_B^+}(f(x)f(y)) = \mu_B^+(xy)$ for all $x, y \in V$. Hence, for $x, y \in V$ we obtain

$$\mu_B^-(xy) = \overline{\mu^-}_B(f(x)f(y)) = \min(\mu_A^-(f(x)), \mu_A^-(f(y))) = \min(\mu_A^-(x), \mu_A^-(y)),$$

which implies *a*). The proof of *b*) is analogous. □

Proposition 5.8. *Let $G = (A, B)$ be an interval-valued fuzzy complete graph. If $\mu_B^-(xy) = \min(\mu_A^-(x), \mu_A^-(y))$ and $\mu_B^+(xy) = \min(\mu_A^+(x), \mu_A^+(y))$ for all x, $y \in V$, then G is self complementary.*

Proof. Let $G = (A, B)$ be an interval-valued fuzzy complete graph such that $\mu_B^-(xy) = \min(\mu_A^-(x), \mu_A^-(y))$ and $\mu_B^+(xy) = \min(\mu_A^+(x), \mu_A^+(y))$ for all x, $y \in V$. Then $G = \overline{G}$ under the identity map $I : V \to V$. So $\overline{\overline{G}} = G$. Hence G is self complementary. □

Proposition 5.9. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be interval-valued fuzzy complete graphs. Then $G_1 \cong G_2$ if and only if $\overline{G}_1 \cong \overline{G}_2$.*

Proof. Assume that G_1 and G_2 are isomorphic, there exists a bijective map $f : V_1 \to V_2$ satisfying

$$\mu_{A_1}^-(x) = \mu_{A_2}^-(f(x)),\ \mu_{A_1}^+(x) = \mu_{A_2}^+(f(x)) \text{ for all } x \in V_1,$$
$$\mu_{B_1}^-(xy) = \mu_{B_2}^-(f(x)f(y)),\ \mu_{B_1}^+(xy) = \mu_{B_2}^+(f(x)f(y)) \text{ for all } xy \in E_1.$$

By definition of complement, we have

$$\overline{\mu^-}_{B_1}(xy) = \min(\mu_{A_1}^-(x), \mu_{A_1}^-(y) = \min(\mu_{A_2}^-(f(x)), \mu_{A_2}^-(f(y))) = \overline{\mu^-}_{B_2}(f(x)f(y)),$$

$\overline{\mu^+}_{B_1}(xy) = \min(\mu_{A_1}^+(x), \mu_{A_1}^+(y) = \min(\mu_{A_2}^+(f(x)), \mu_{A_2}^+(f(y))) = \overline{\mu^+}_{B_2}(f(x)f(y))$ for all $xy \in E_1$. Hence $\overline{G}_1 \cong \overline{G}_2$.

The proof of converse part is straightforward. □

6 Conclusions

It is well known that interval-valued fuzzy sets constitute a generalization of the notion of fuzzy sets. The interval-valued fuzzy models give more precision, flexibility and compatibility to the system as compared to the classical and fuzzy models. So we have introduced interval-valued fuzzy graphs and have presented several properties in this paper. The further study of interval-valued fuzzy graphs may also be extended with the following projects:

- an application of interval-valued fuzzy graphs in database theory
- an application of interval-valued fuzzy graphs in an expert system
- an application of interval-valued fuzzy graphs in neural networks
- an interval-valued fuzzy graph method for finding the shortest paths in networks

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